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L^p MULTIPLIERS WITH WEIGHT $|x|^{kp-1}$

BY
BENJAMIN MUCKENHOUPT[1] AND WO-SANG YOUNG[2]

ABSTRACT. Let k be a positive integer and $1 < p < \infty$. It is shown that if T is a multiplier operator on L^p of the line with weight $|x|^{kp-1}$, then Tf equals a constant times f almost everywhere. This does not extend to the periodic case since $m(j) = 1/j$, $j \neq 0$, is a multiplier sequence for L^p of the circle with weight $|x|^{kp-1}$. A necessary and sufficient condition is derived for a sequence $m(j)$ to be a multiplier on L^2 of the circle with weight $|x|^{2k-1}$.

1. Introduction. This is a continuation of our previous work with R. L. Wheeden on multipliers for weighted L^p spaces [**2**, **3**]. The underlying space we consider here will either be the line $\mathbf{R}$ or the circle $[-\pi, \pi]$; $\hat{f}$ and $\check{f}$ will denote the Fourier transform and inverse Fourier transform of f. For $1 < p < \infty$, L^p_γ is the space of functions f with $\| f \|_{p,\gamma} = (\int |f|^p |x|^\gamma dx)^{1/p} < \infty$. Let $\mathcal{S}_{00}$ be the set of all functions whose Fourier transforms are infinitely differentiable, have compact support and vanish near the origin. A function m, locally integrable in $\mathbf{R} \setminus \{0\}$, is said to be a multiplier for $L^p_\gamma(\mathbf{R})$ if

$$\int |(m\hat{f})^{\vee}|^p |x|^\gamma dx \leq c \int |f|^p |x|^\gamma dx \tag{1.1}$$

for all f in $\mathcal{S}_{00}$. There is no analogue of $\mathcal{S}_{00}$ for the periodic case. Instead, for $k = 0, 1, 2, \ldots$, we let S_k be the set of trigonometric polynomials with $\hat{f}(j) = 0$, $j = 0, 1, \ldots, k$. A sequence of numbers $\{m(j)\}_{j=-\infty}^{\infty}$ is a multiplier for $L^p_\gamma([-\pi, \pi])$ if (1.1) holds for f in S_k, $k = [(\gamma + 1)/p] - 1$. (As usual, $[\alpha]$ is the largest integer less than or equal to α.)

In [**2**] we characterized multipliers for L^2_γ, $\gamma \neq 2k - 1$, k a positive integer. Results for the line and the circle are analogues of each other. In [**3**] we studied weighted L^p results for various classes of multipliers. In particular, sufficient conditions were obtained for a function $m(x)$ to be a multiplier for L^p_γ for all $\gamma > -1$ if $\gamma \neq kp - 1$, k a positive integer. In this paper we consider multipliers for the "missing indices", $\gamma = kp - 1$, k a positive integer. Unlike [**2** and **3**], the results obtained here are different for the line and the circle. We will show in §2 that multipliers on $L^p_{kp-1}(\mathbf{R})$ are trivial by proving the following theorem.

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THEOREM (1.2). *Let k be a positive integer, $1 < p < \infty$, and m be locally integrable in $\mathbf{R} \setminus \{0\}$. If for all f in $\mathcal{S}_{00}$,*

$$\int_{\mathbf{R}} |(m\hat{f})\check{}|^p |x|^{kp-1}\,dx \leqslant c\int_{\mathbf{R}} |f|^p |x|^{kp-1}\,dx \tag{1.3}$$

with c independent of f, then m equals a constant a.e.

In contrast to this theorem we will now show that $m(j) = 1/j$, $j \neq 0$, is a multiplier for $L^p_{kp-1}([-\pi, \pi])$, $1 < p < \infty$, k a positive integer. Let f be a trigonometric polynomial in S_{k-1}. Then

$$i(m\hat{f})\check{}(x) = \int_x^{\pi} f(t)\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} tf(t)\,dt.$$

By Hardy's inequality [5, p. 272] and the fact that $|x|^p < c$ if $|x| < \pi$,

$$\int_{-\pi}^{\pi}\left|\int_x^{\pi} f(t)\,dt\right|^p |x|^{kp-1}\,dx \leqslant c\int_{-\pi}^{\pi} |f|^p |x|^{kp-1}\,dx.$$

To estimate the integral of the second term, we observe that, since $f \in S_{k-1}$,

$$\left|\int_{-\pi}^{\pi} tf(t)\,dt\right| = \left|\int_{-\pi}^{\pi}\left(t - \sum_{j=1}^{k-1}\frac{(1-e^{-it})^j}{ij}\right)f(t)\,dt\right|. \tag{1.4}$$

The fact that $\sum_{j=1}^{k-1} z^j/j = -\log(1-z) + O(z^k)$ shows that the right side of (1.4) is bounded by $c\int_{-\pi}^{\pi}|t|^k|f(t)|\,dt$. Hence, we have

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} tf(t)\,dt\right|^p |x|^{kp-1}\,dx &\leqslant c\left(\int_{-\pi}^{\pi}|t|^k|f(t)|dt\right)^p \\ &\leqslant c\int_{-\pi}^{\pi}|t|^{kp-1}|f(t)|^p\,dt,\end{aligned}$$

by Hölder's inequality. Therefore $m(j) = 1/j$ is a multiplier for $L^p_{kp-1}([-\pi, \pi])$.

In §§3–5, we characterize all multipliers for $L^2_{2k-1}([-\pi, \pi])$, k a positive integer. In the following Δ denotes the forward difference, i.e., $\Delta^0 m(j) = m(j)$, and for $s = 0, 1, 2, \ldots$, $\Delta^{s+1}m(j) = \Delta^s m(j+1) - \Delta^s m(j)$. Also, for any integer l and for $s = 0, 1, 2, \ldots, l^{(s)}$ is defined by $l^{(s)} = l(l-1)(l-2)\cdots(l-s+1)$ for $s > 1$ and $l^{(0)} = 1$. The characterization is given in Theorem (1.5).

THEOREM (1.5). *Let k be a positive integer and $\{m(j)\}_{j=-\infty}^{\infty}$ be a sequence of numbers with $m(j) = 0$ for $0 < j < k-1$. Then*

$$\int_{-\pi}^{\pi} |(m\hat{f})\check{}|^2 |x|^{2k-1}\,dx \leqslant c\int_{-\pi}^{\pi} |f|^2 |x|^{2k-1}\,dx \tag{1.6}$$

for all $f \in S_{k-1}$ if and only if $m \in l^\infty$, and, for $h = 1, 2, \ldots, k$, m satisfies

$$\sum_{|j|>\mu} j^{2(h-1)} \sum_{l\neq 0} \frac{\left|m(j+l) - \sum_{s=0}^{h-1}(l(s)/s!)\Delta^s m(j)\right|^2}{l^{2h}} \leqslant \frac{c}{\log(\mu+1)}, \tag{1.7}$$

$\mu = 1, 2, \ldots.$

In this paper, c denotes a positive constant which depends on k and m and may vary from line to line.

2. The nonperiodic case. To prove Theorem (1.2) we first observe that (1.3) implies that $m \in L^\infty$. This can be shown in exactly the same way as in the proof of Theorem (2.2) in [**2**]. The rest of the proof is contained in the following lemmas.

LEMMA (2.1). *Let $m \in L^\infty$, $1 < p < \infty$, and $k = 1, 2, \ldots$. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of functions in $\mathbb{S}_{00}$ such that for sufficiently large n,*

$$\int_{\mathbf{R}} \left|\left(m\hat{f}_n\right)^\vee\right|^p |x|^{kp-1}\, dx \leqslant c \log n, \tag{2.2}$$

$$\int_{\mathbf{R}} |f_n|^p |x|^{kp-1}\, dx \leqslant c \log n, \tag{2.3}$$

and

(2.4) *there is a subset $A \subset [1,2]$ with positive measure such that $c\,|\hat{f}_n(x)| \geqslant \log n$ for $x \in A$. Then $m =$ constant a.e.*

LEMMA (2.5). *Let $1 < p < \infty$, and $k = 1, 2, \ldots$. Then there exists a sequence of functions $\{f_n\}_{n=1}^\infty$ in $\mathbb{S}_{00}$ such that* (2.3) *and* (2.4) *hold for sufficiently large n.*

PROOF OF LEMMA (2.1). We first consider the case $k = 1$. The inequality

$$\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{|\hat{f}(x+h) - \hat{f}(x)|^p}{|h|^p} |x|^{p-2}\, dh\, dx \leqslant c\int_{\mathbf{R}} |f|^p |x|^{p-1}\, dx \tag{2.6}$$

is proved in [**5**, pp. 139–140] for $p = 2$. The same proof, using a continuous version of Paley's theorem [**6**, Volume II, p. 121] instead of Plancherel's formula, implies (2.6) for $1 < p \leqslant 2$. Similarly,

$$\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{|\hat{f}(x+h) - \hat{f}(x)|^p}{|h|^2}\, dh\, dx \leqslant c\int_{\mathbf{R}} |f|^p |x|^{p-1}\, dx, \qquad 2 \leqslant p < \infty. \tag{2.7}$$

We shall consider the case $1 < p \leqslant 2$. The case $2 < p < \infty$ can be done in the same way. By (2.6) and Minkowski's inequality, we have

(2.8)

$$\int_{\mathbf{R}} \left|\left(m\hat{f}_n\right)^\vee\right|^p |x|^{p-1}\, dx \geqslant c\int_A \left[\int_{\mathbf{R}} \frac{|m(x+h) - m(x)|^p}{|h|^p}\, dh\right] |\hat{f}_n(x)|^p |x|^{p-2}\, dx$$
$$-c\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{|m(x+h)|^p\, |\hat{f}_n(x+h) - \hat{f}_n(x)|^p\, |x|^{p-2}}{|h|^p}\, dh\, dx.$$

The left side is less than $c \log n$ by (2.2). The first term on the right is larger than

$$c\int_A\int_{\mathbf{R}} \frac{|m(x+h) - m(x)|^p}{|h|^p}\, dh\, dx (\log n)^p$$

by (2.4). Using the boundedness of m, (2.6) and (2.3), we see that the second term on the right has absolute value less than $c \log n$. Combining all these, we have

$$\int_A\int_{\mathbf{R}} \frac{|m(x+h) - m(x)|^p}{|h|^p}\, dh\, dx \leqslant c(\log n)^{1-p}$$

for all sufficiently large n. This implies that

$$\int_A \int_{\mathbf{R}} \frac{|m(x+h)-m(x)|^p}{|h|^p}\,dh\,dx = 0,$$

which in turn implies that $m(x) =$ constant a.e.

To prove Lemma (2.1) for the case $k > 1$, we shall show that there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{S}_{00}$ satisfying (2.2)–(2.4) with k replaced by $k-1$. The lemma will then follow by induction.

To do this, we define $g_n(x) = \int_x^\infty f_n(t)\,dt$. Then, since $g_n(x) = -\int_{-\infty}^x f_n(t)\,dt$, it is easy to verify that $g_n \in L^1$ and $\hat{g}_n(x) = -(1/ix)\hat{f}_n(x)$, $x \neq 0$. Hence $g_n \in \mathcal{S}_{00}$, and by (2.4),

$$|\hat{g}_n(x)| = \frac{1}{|x|}|\hat{f}_n(x)| \geqslant c\log n$$

for $x \in A$. To show that $\{g_n\}$ satisfies (2.2) for $k-1$, we observe that

$$\begin{aligned}\int_{\mathbf{R}} |(m\hat{g}_n)^{\vee}|^p |x|^{(k-1)p-1}\,dx &= \int_{\mathbf{R}} \left|\left(\frac{1}{-ix}\, m\hat{f}_n\right)^{\vee}\right|^p |x|^{(k-1)p-1}\,dx \\ &= \int_{\mathbf{R}} \left|\int_x^\infty (m\hat{f}_n)^{\vee}\,dt\right|^p |x|^{(k-1)p-1}\,dx.\end{aligned}$$

Hardy's inequality [5, p. 272] implies that the last integral is less than

$$c\int_{\mathbf{R}} |(m\hat{f}_n)^{\vee}|^p |x|^{kp-1}\,dx,$$

which in turn is less than $c\log n$ by (2.2). The proof that $\{g_n\}$ satisfies (2.3) for $k-1$ is similar.

This completes the proof of Lemma (2.1).

PROOF OF LEMMA (2.5). We first show that there is a sequence $\{g_{n,k}\}_{n=1}^{\infty}$ in L^∞ with compact support such that for sufficiently large n,

(2.9) $$\hat{g}_{n,k}(0) = \hat{g}'_{n,k}(0) = \cdots = \hat{g}_{n,k}^{(k-1)}(0) = 0,$$

(2.10) $$\int_{\mathbf{R}} |g_{n,k}|^p |x|^{kp-1}\,dx \leqslant c\log n,$$

and

(2.11) $$|\hat{g}_{n,k}(x)| \geqslant c\log n \quad \text{for } x \in [1,2].$$

The $g_{n,k}$'s are defined as follows. For $n = 1, 2, \ldots,$ let

$$g_{n,1}(x) = x^{-1}\left[\chi_{[1/n,1]}(x) - \chi_{[1,n]}(x)\right].$$

For $k > 1$, $g_{n,k}$ is defined recursively by

(2.12) $$g_{n,k}(x) = x^{-1}\left[g_{n,k-1}(x) - g_{n,k-1}(2x)\right].$$

We can also write $g_{n,k}(x)$, $k > 1$, as

(2.13) $$g_{n,k}(x) = \frac{1}{x^{k-1}} \sum_{j=0}^{k-1} c_{k,j} g_{n,1}(2^j x),$$

where $c_{k,j}$ is given by

$$\sum_{j=0}^{k-1} c_{k,j} x^j = \prod_{l=0}^{k-2}\left(1 - \frac{x}{2^l}\right). \tag{2.14}$$

That $\{g_{n,k}\}$ satisfies (2.9) and (2.10) follows readily from (2.12) and induction on k. The proof of (2.11) is divided into two cases: $k = 1$ and $k > 1$. For $k = 1$, we have

$$|\hat{g}_{n,1}(x)| \geq \left|\int_{1/n}^{1} \frac{1}{t}\,dt\right| - \left|\int_{1/n}^{1} \frac{1}{t}(e^{-ixt} - 1)\,dt\right| - \left|\int_{1}^{n} \frac{1}{t} e^{-ixt}\,dt\right|.$$

The first term on the right is equal to $\log n$. The second is majorized by $\int_{1/n}^{1} xt/t\,dt$, which is less than 2 for $x \in [1,2]$. By integration by parts, the third term is equal to

$$\left|\frac{e^{-ixn}}{-ixn} - \frac{e^{-ix}}{-ix} - \int_{1}^{n} \frac{e^{-ixt}}{ixt^2}\,dt\right|.$$

This is less than $1/nx + 1/x + (1/x)(1 - 1/n) \leq 2$, $x \in [1,2]$. Hence, for sufficiently large n, $|\hat{g}_{n,1}(x)| \geq c\log n$, $x \in [1,2]$.

For $k > 1$, we recall that $\hat{g}_{n,k}^{(l)}(0) = 0$, $l = 0, 1, \ldots, k-2$, and write

$$\begin{aligned}
\hat{g}_{n,k}(x) &= \hat{g}_{n,k}(x) - \sum_{l=0}^{k-2} \frac{x^l}{l!}\hat{g}_{n,k}^{(l)}(0)\\
&= \int_R g_{n,k}(t)\left[e^{-ixt} - \sum_{l=0}^{k-2} \frac{(-it)^l}{l!}x^l\right] dt.
\end{aligned}$$

Using (2.13) and a change of variables, we get

$$\begin{aligned}
\hat{g}_{n,k}(x) &= \sum_{j=0}^{k-1} c_{k,j} 2^{j(k-2)} \int_{\mathbf{R}} \frac{1}{t^{k-1}} g_{n,1}(t)\left[e^{-ix2^{-j}t} - \sum_{l=0}^{k-2} \frac{(-ix2^{-j}t)^l}{l!}\right] dt \\
&= \sum_{j=0}^{k-1} c_{k,j} 2^{j(k-2)} \int_{1/n}^{1} \frac{1}{t^k} \frac{(-ix2^{-j}t)^{k-1}}{(k-1)!}\,dt \\
&\quad + \sum_{j=0}^{k-1} c_{k,j} 2^{j(k-2)} \int_{1/n}^{1} \frac{1}{t^k}\left[e^{-ix2^{-j}t} - \sum_{l=0}^{k-1} \frac{(-ix2^{-j}t)^l}{l!}\right] dt \\
&\quad - \sum_{j=0}^{k-1} c_{k,j} 2^{j(k-2)} \int_{1}^{n} \frac{1}{t^k}\left[e^{-ix2^{-j}t} - \sum_{l=0}^{k-2} \frac{(-ix2^{-j}t)^l}{l!}\right] dt.
\end{aligned} \tag{2.15}$$

For $x \in [1,2]$, the modulus of the first term on the right is larger than

$$\frac{1}{(k-1)!}\left|\sum_{j=0}^{k-1} c_{k,j} 2^{-j}\right| \log n.$$

This, because of (2.14), is larger than $c\log n$ for some $c > 0$. To estimate the second term on the right of (2.15) we observe that

$$\left|e^{-ix2^{-j}t} - \sum_{l=0}^{k-1} \frac{(-ix2^{-j}t)^l}{l!}\right| \leq \left|\frac{(-ix2^{-j}t)^k}{k!}\right|.$$

Hence the modulus of the second term is majorized by c, $x \in [1,2]$. Finally, estimating the integral of the terms separately, we obtain that the modulus of the last term of (2.15) is less than c, $x \in [1,2]$. Hence, when n is sufficiently large, $|\hat{g}_{n,k}(x)| \geq c \log n$, $x \in [1,2]$.

We have thus obtained a sequence $\{g_{n,k}\}_{n=1}^{\infty}$ that satisfies (2.9)–(2.11). To finish the proof of the lemma, all we have to do is approximate each $g_{n,k}$ by a function $f_n \in \mathbb{S}_{00}$ in both the L^2 norm and L^p_{kp-1} norm. This is possible because of Theorem (6.1) of **[2]**. This completes the proof of Lemma (2.5) and hence the proof of Theorem (1.2).

3. Periodic case: preliminaries. The proof of Theorem (1.5) is more difficult, partly because there are multipliers for the periodic case, and partly because of the complications that result from replacing derivatives by differences. We shall need the following lemmas.

LEMMA (3.1). *Let $k = 1, 2, \ldots$, let f be a trigonometric polynomial. Then*

$$\int_{-\pi}^{\pi} |f(x)|^2 |x|^{2k-1} \, dx \approx \sum_j \sum_{l \neq 0} \frac{\left|\hat{f}(j+l) - \sum_{s=0}^{k-1} (l^{(s)}/s!) \Delta^s \hat{f}(j)\right|^2}{l^{2k}}, \tag{3.2}$$

where, as usual, $\approx$ means that each side is bounded above by c times the other side.

PROOF. We first observe that for $s = 0, 1, \ldots$,

$$\Delta^s \hat{f}(j) = \sum_{h=0}^{s} (-1)^{s-h} \binom{s}{h} \hat{f}(j+h) = \int_{-\pi}^{\pi} f(t)(e^{-it} - 1)^s e^{-ijt} \, dt. \tag{3.3}$$

Hence the right side of (3.2) is equal to

$$\sum_{l \neq 0} \frac{1}{l^{2k}} \sum_j \left| \int_{-\pi}^{\pi} f(t) \left[e^{-ilt} - \sum_{s=0}^{k-1} \frac{l^{(s)}}{s!} (e^{-it} - 1)^s \right] e^{-ijt} \, dt \right|^2,$$

which, by Parseval's identity, is equal to

$$c \int_{-\pi}^{\pi} |f(t)|^2 \sum_{l \neq 0} \frac{\left| e^{-ilt} - \sum_{s=0}^{k-1} (l^{(s)}/s!)(e^{-it} - 1)^s \right|^2}{l^{2k}} \, dt.$$

The lemma will be proved if we show that

$$\sum_{l \neq 0} \frac{\left| e^{-ilt} - \sum_{s=0}^{k-1} (l^{(s)}/s!)(e^{-it} - 1)^s \right|^2}{l^{2k}} \approx |t|^{2k-1}. \tag{3.4}$$

To prove (3.4), we let $\phi(z) = z^l$. Repeated integration by parts, as in the proof of Taylor's theorem for a real variable, shows that for $z = e^{-it}$, $|t| \leq \pi$,

$$\phi(z) = \sum_{s=0}^{k-1} \frac{\phi^{(s)}(1)}{s!} (z-1)^s + \int_{\Gamma} \frac{\phi^{(k)}(\zeta)}{(k-1)!} (\zeta - 1)^{k-1} \, d\zeta,$$

where Γ is the path on the unit circle joining 1 to z. From this it follows that for l an integer,

$$(3.5)\qquad e^{-ilt}-\sum_{s=0}^{k-1}\frac{l^{(s)}}{s!}(e^{-it}-1)^s=\int_\Gamma\frac{l^{(k)}}{(k-1)!}\zeta^{l-k}(\zeta-1)^{k-1}\,d\zeta$$
$$=-i\frac{l^{(k)}}{(k-1)!}\int_0^t e^{-il\theta}(1-e^{i\theta})^{k-1}\,d\theta.$$

Hence

$$\left|e^{-ilt}-\sum_{s=0}^{k-1}\frac{l^{(s)}}{s!}(e^{-it}-1)^s\right|\leqslant c\frac{|l^{(k)}|}{k!}|t|^k\leqslant c|l|^k|t|^k.$$

Therefore

$$\sum_{0<|l|<1/|t|}\frac{\left|e^{-ilt}-\sum_{s=0}^{k-1}(l^{(s)}/s!)(e^{-it}-1)^s\right|^2}{l^{2k}}\leqslant c|t|^{2k-1}.$$

On the other hand, we have

$$\left|e^{-ilt}-\sum_{s=0}^{k-1}\frac{l^{(s)}}{s!}(e^{-it}-1)^s\right|\leqslant\left|e^{-ilt}-\sum_{s=0}^{k-2}\frac{l(s)}{s!}(e^{-it}-1)^s\right|$$
$$+\frac{|l^{(k-1)}|}{(k-1)!}|e^{-it}-1|^{k-1}$$
$$\leqslant c|l|^{k-1}|t|^{k-1}.$$

Thus

$$\sum_{|l|\geqslant 1/|t|}\frac{\left|e^{-ilt}-\sum_{s=0}^{k-1}(l^{(s)}/s!)(e^{-it}-1)^s\right|^2}{l^{2k}}\leqslant c|t|^{2k-1}.$$

We have proved the inequality "$\leqslant$" in (3.4).

To prove the reverse inequality in (3.4), we consider two cases: $|t|<\frac{1}{4}(2/\pi)^k$ and $|t|\geqslant\frac{1}{4}(2/\pi)^k$. For $|t|<\frac{1}{4}(2/\pi)^k$, it is enough to restrict the sum in (3.4) to $-\frac{1}{4}(2/\pi)^k t^{-1}<l<0$. For $l<0$, $|l^{(k)}|/k!\geqslant(1/k!)|l|^k$. Also,

$$\left|\int_0^t e^{-il\theta}(1-e^{i\theta})^{k-1}\,d\theta\right|\geqslant\left|\int_0^t(1-e^{i\theta})^{k-1}e^{i\theta}\,d\theta\right|$$
$$-\left|\int_0^t(e^{-i(l+1)\theta}-1)(1-e^{i\theta})^{k-1}e^{i\theta}\,d\theta\right|.$$

The first term on the right is equal to $|1-e^{it}|^k/k$, which is larger than $(2/\pi)^k|t|^k/k$ for $|t|<\pi$. The second term on the right is less than $(|t|^{k+1}/(k+1))|l+1|$, which is less than $\frac{1}{2}(2/\pi)^k|t|^k/k$ for $0>l>-\frac{1}{4}(2/\pi)^k|t|^{-1}$. Using (3.5), we obtain, for $-\frac{1}{4}(2/\pi)^k|t|^{-1}<l<0$,

$$C\left|e^{-ilt}-\sum_{s=0}^{k-1}\frac{l^{(s)}}{s!}(e^{-it}-1)^s\right|\geqslant|l|^k|t|^k,$$

and so

$$\sum_{-(1/4)(2/\pi)^k(1/|t|)<l<0} \frac{C\left|e^{-ilt} - \sum_{s=0}^{k-1}(l^{(s)}/s!)(e^{-it}-1)^s\right|^2}{l^{2k}} \geqslant |t|^{2k-1}.$$

For the case $|t| \geqslant \frac{1}{4}(2/\pi)^k$, we only have to take the term $l = -1$. Since $|(-1)^{(k)}|/k! = 1$, and

$$\left|\int_0^t e^{-i(-1)\theta}(1-e^{i\theta})^{k-1}\,d\theta\right| = \frac{|1-e^{it}|^k}{k} \geqslant \left(\frac{2}{\pi}\right)^k \frac{|t|^k}{k}, \qquad |t| \leqslant \pi,$$

(3.5) implies that the left side of (3.4) is bounded below by $(2/\pi)^{2k}(|t|^{2k}/k^2)$. This in turn is larger than $C|t|^{2k-1}$ for $\frac{1}{4}(2/\pi)^k \leqslant |t| \leqslant \pi$. This completes the proof of Lemma (3.1).

The second lemma deals with products.

LEMMA (3.6). *Let $k = 1, 2, \ldots$. For any two functions f, g defined on the integers, and any integers j and l,*

$$\begin{aligned}(fg)(j+l) - \sum_{s=0}^{k-1}\frac{l^{(s)}}{s!}\Delta^s(fg)(j) &= f(j+l)\left[g(j+l) - \sum_{s=0}^{k-1}\frac{l^{(s)}}{s!}\Delta^s g(j)\right] \\ &+ \sum_{h=0}^{k-1}\left[f(j+l) - \sum_{s=0}^{k-h-1}\frac{(l-h)^{(s)}}{s!}\Delta^s f(j+h)\right]\frac{l^{(h)}}{h!}\Delta^h g(j).\end{aligned}$$

PROOF. This identity can be verified using induction and Leibnitz's rule for products:

$$\Delta^k(fg)(j) = \sum_{h=0}^{k}\binom{k}{h}\Delta^h g(j)\Delta^{k-h}f(j+h), \tag{3.7}$$

where j is an integer, and k is a nonnegative integer.

The last lemma relates the L^2_{2k-1} norm of $(m\hat{f})\check{}$ to the Fourier coefficients of f.

LEMMA (3.8). *Let $k = 1, 2, \ldots$, $\{m(j)\}_{j=-\infty}^{\infty}$ a sequence of numbers, f be a trigonometric polynomial and A a set of integers. Then*

(3.9)

$$\begin{aligned}\int_{-\pi}^{\pi}\left|(m\hat{f})\check{}\right|^2|x|^{2k-1}\,dx &\leqslant c\sum_j\sum_{l\neq 0}\frac{|m(j+l)|^2\left|\hat{f}(j+l) - \sum_{s=0}^{k-1}(l^{(s)}/s!)\Delta^s\hat{f}(j)\right|^2}{l^{2k}} \\ &+ c\sum_{h=0}^{k-1}\sum_j\sum_{l\neq 0}\frac{\left|m(j+l) - \sum_{s=0}^{k-h-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2k-2h}}\left|\Delta^h\hat{f}(j)\right|^2\end{aligned}$$

and

(3.10)

$$\int_{-\pi}^{\pi}\left|(m\hat{f})^{\check{}}\right|^2|x|^{2k-1}\,dx \geq c\sum_{j\in A}\sum_{l\neq 0}\frac{\left|m(j+l)-\sum_{s=0}^{k-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{e^{2k}}\left|\hat{f}(j)\right|^2$$
$$-c\sum_{h=1}^{k-1}\sum_{j\in A}\sum_{l\neq 0}\frac{\left|m(j+l)-\sum_{s=0}^{k-h-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2k-2h}}\left|\Delta^h\hat{f}(j)\right|^2$$
$$-c\sum_{j\in A}\sum_{l\neq 0}\frac{\left|m(j+l)\right|^2\left|\hat{f}(j+l)-\sum_{s=0}^{k-1}(l^{(s)}/s!)\Delta^s\hat{f}(j)\right|^2}{l^{2k}},$$

where the c's depend only on k.

PROOF. In view of Lemmas (3.1) and (3.6), and the fact that $|l^{(h)}/h!|\leq c|l|^h$ for any integer l and $h=1,\ldots,k-1$, it is sufficient to show that for $h=1,\ldots,k-1$ and any integers j and l,

$$(3.11)\quad \sum_{l\neq 0}\frac{\left|m(j+l)-\sum_{s=0}^{k-h-1}\left((l-h)^{(s)}/s!\right)\Delta^s m(j+h)\right|^2}{l^{2k-2h}}$$
$$\leq c\sum_{l\neq 0}\frac{\left|m(j+l)-\sum_{s=0}^{k-h-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2k-2h}}.$$

To prove (3.11), define

$$(3.12)\qquad R(j,l,k,h)=\sum_{s=0}^{k-h-1}\frac{(l-h)^{(s)}}{s!}\Delta^s m(j+h)-\frac{l^{(s)}}{s!}\Delta^s m(j).$$

The proof of (3.11) will be completed by showing that

$$(3.13)\quad \sum_{l\neq 0}\frac{\left|R(j,l,k,h)\right|^2}{l^{2k-2h}}\leq c\sum_{l\neq 0}\frac{\left|m(j+l)-\sum_{s=0}^{k-h-1}(l^{(s)}/s!)\Delta^s m(h)\right|^2}{2^{2k-2h}}.$$

To prove (3.13) we will first prove the identity

(3.14)

$$R(j,l,k,h)=\sum_{t=0}^{k-h-1}\frac{(l-h)^{(t)}}{t!}\sum_{q=0}^{t}(-1)^{t-q}\binom{t}{q}\left[m(j+h+q)-\sum_{s=0}^{k-h-1}\frac{(h+q)^{(s)}}{s!}\Delta^s m(j)\right].$$

To do this use the expansion in (3.3) on the term $\Delta^s m(j+h)$ in (3.12) to show that

(3.15)

$$R(j,l,k,h)=\sum_{s=0}^{k-h-1}\left[\frac{-l^{(s)}}{s!}\Delta^s m(j)+\frac{(l-h)^{(s)}}{s!}\sum_{q=0}^{s}(-1)^{s-q}\binom{s}{q}m(j+h+q)\right].$$

The difference of the right side of (3.14) and the right side of (3.15) is

$$\sum_{s=0}^{k-h-1}\left[l^{(s)}-\sum_{t=0}^{k-h-1}\frac{(l-h)^{(t)}}{t!}\sum_{q=0}^{t}(-1)^{t-q}\binom{t}{q}(h+q)^{(s)}\right]\frac{\Delta^s m(j)}{s!};$$

to show that this is 0 it is enough to prove that

$$l^{(s)}=s!\sum_{t=0}^{k-h-1}\frac{(l-h)^{(t)}}{t!}\sum_{q=0}^{t}(-1)^{t-q}\binom{t}{q}\binom{h+q}{s} \tag{3.16}$$

for $0 \leqslant s \leqslant k-h-1$.By induction on t or by (8) on p. 10 of **[4]**, the inner sum equals $\binom{h}{s-t}$. Therefore, the right side of (3.16) equals

$$s!\sum_{t=0}^{k-h-1}\binom{l-h}{t}\binom{h}{s-t}. \tag{3.17}$$

Since $0 \leqslant s \leqslant k-h-1$, Vandermonde's identity shows that (3.17) equals $s!\binom{l}{s} = l^{(s)}$. This completes the proof of (3.16) and establishes (3.14).

To prove (3.13), change the order of summation in (3.14) to show that for $l \neq 0$,

$$R(j,l,k,h) \leqslant c\sum_{q=0}^{k-h-1}\left|m(j+h+q)-\sum_{s=0}^{k-h-1}\frac{(h+q)^{(s)}}{s!}\Delta^s m(j)\right||l|^{k-h-1}.$$

Hence

$$\sum_{l\neq 0}\frac{|R(j,l,k,h)|^2}{l^{2k-2h}} \leqslant c\sum_{q=0}^{k-h-1}\sum_{l\neq 0}\frac{1}{l^2}\left|m(j+h+q)-\sum_{s=0}^{k-h-1}\frac{(h+q)^{(s)}}{s!}\Delta^s m(j)\right|^2$$

$$\leqslant c\sum_{q=0}^{k-h-1}\frac{\left|m(j+h+q)-\sum_{s=0}^{k-h-1}\left((h+q)^{(s)}/s!\right)\Delta^s m(j)\right|^2}{|h+q|^{2k-2h}},$$

which is less than the right side of (3.13). This completes the proof of Lemma (3.8).

4. Periodic case: necessity. Suppose $m(j)=0$ for $0 \leqslant j \leqslant k-1$ and (1.6) holds for all $f \in S_{k-1}$. We first observe that $\{m(j)\} \in l^\infty$. This can be seen by taking $f(x)=e^{ijx}$, $j \leqslant -1$ and $j \geqslant k$, in (1.6). The rest of the proof is based on the following lemmas.

LEMMA (4.1). *Let $k=1,2,\ldots,$ and $\{m(j)\}_{j=-\infty}^{\infty} \in l^\infty$. Suppose there is a trigonometric polynomial f, a positive constant $B=B(f)$, a constant D depending only on k, and a nonempty set of nonzero integers A such that*

$$\hat{f}(j)=0, \qquad j=0,1,\ldots,k-1, \tag{4.2}$$

$$\int_{-\pi}^{\pi}|(m\hat{f})\check{}\,|^2|x|^{2k-1}\,dx \leqslant DB, \tag{4.3}$$

$$\int_{-\pi}^{\pi}|f|^2|x|^{2k-1}\,dx \leqslant DB, \tag{4.4}$$

$$|\hat{f}(j)| \geqslant B|j|^{k-1}, \qquad j \in A, \tag{4.5}$$

and

$$|\Delta^h \hat{f}(j)| \leqslant DB|j|^{k-h-1}, \qquad h = 0,1,\ldots,k-1, j \in A. \tag{4.6}$$

Then, for $h = 1,2,\ldots,k$, m satisfies

$$\sum_{j\in A} j^{2(h-1)} \sum_{l\neq 0} \frac{\left|m(j+l) - \sum_{s=0}^{h-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2h}} \leqslant \frac{c}{B}, \tag{4.7}$$

where c depends only on k and $\|m\|_\infty$.

This will be used with functions given by the next two lemmas.

LEMMA (4.8). *Let $k = 1,2,\ldots$. Then there exists an integer $N \geqslant 2k$, a number $1 < r < 2$, and a sequence of trigonometric polynomials $\{f_n\}_{n=1}^\infty$ such that for $n \geqslant N$,* (4.2), (4.4)–(4.6) *are satisfied with $A = \{j\colon n \leqslant |j| < n^r\}$ and $B = c\log n$, where c is positive and independent of n.*

LEMMA (4.9). *Let $k = 1,2,\ldots,$ and $N \geqslant 2k$ be an integer. Then there exists a trigonometric polynomial f such that* (4.2), (4.4)–(4.6) *are satisfied with $B > 0$ and $A = \{j\colon 1 \leqslant |j| \leqslant N\}$.*

Suppose we have these three lemmas, $m(j) = 0$ for $0 \leqslant j \leqslant k-1$ and m satisfies (1.6) for f in S_{k-1}. Let N and n be as in Lemma (4.8). Given $\mu \geqslant N$, let t be a nonnegative integer and let $n = [\mu^{r^t}]$. Because of (1.6), the polynomials f_n of Lemma (4.8) satisfy (4.2)–(4.6) with $B = c\log n$ and D independent of n. Therefore, by Lemma (4.1),

$$\sum_{\mu^{r^t}\leqslant |j| < \mu^{r^{t+1}}} j^{2(h-1)} \sum_{l\neq 0} \frac{\left|m(j+l) - \sum_{s=0}^{h-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2h}} \leqslant \frac{c}{r^t \log\mu}.$$

Adding these for $t = 0,1,2,\ldots$ proves (1.7) for $\mu \geqslant N$. Similarly, from Lemma (4.9), (1.6) and Lemma (4.1) we have

$$\sum_{1\leqslant |j| \leqslant N} j^{2(h-1)} \sum_{l\neq 0} \frac{\left|m(j+l) - \sum_{s=0}^{h-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2h}} \leqslant c.$$

Combining this with the previous case proves (1.7) for all $\mu \geqslant 1$.

PROOF OF LEMMA (4.1). The proof for the case $k = 1$ uses Lemma (3.1) to prove a discrete version of (2.8). The reasoning parallels that used in the proof of Lemma (2.1) except for the conclusion.

For $k > 1$, the proof is by induction on k and is given in two steps. First we show that there is a trigonometric polynomial g that satisfies (4.2)–(4.6) with k replaced by $k-1$, with $B(g) = cB(f)$ and $A(g) = A(f)$. From this it follows that m satisfies (4.7) for $h = 1,\ldots,k-1$.

We define g as follows:

$$g(x) = \int_x^\pi f(t)\,dt - \frac{1}{2\pi}\int_{-\pi}^\pi tf(t)\,dt.$$

Then $g \in L^1$, $\hat{g}(0) = 0$, and, for $j \neq 0$,

$$\hat{g}(j) = -\hat{f}(j)/ij.$$

Therefore g satisfies (4.2). To prove (4.3), we observe that

$$(4.10)\quad \int_{-\pi}^{\pi} |(m\hat{g})\check{}|^2 |x|^{2(k-1)-1}\,dx$$

$$= \int_{-\pi}^{\pi} \left|\left(\frac{1}{-ij} m\hat{f}\right)\check{}\right|^2 |x|^{2(k-1)-1}\,dx < c\int_{-\pi}^{\pi} \left|\int_x^{\pi} (m\hat{f})\check{}\,dt\right|^2 |x|^{2(k-1)-1}\,dx$$

$$+ c\int_{-\pi}^{\pi} \left|\int_{-\pi}^{\pi} t(m\hat{f})\check{}\,dt\right|^2 |x|^{2(k-1)-1}\,dx.$$

By Hardy's inequality [5], the first term on the right is less than

$$(4.11)\qquad c\int_{-\pi}^{\pi} |(m\hat{f})\check{}|^2 |x|^{2k-1}\,dx \leqslant cB(f).$$

The argument used to estimate the left side of (1.4) and its integral shows that the second term on the right of (4.10) is also less than (4.11). Hence g satisfies (4.3). A similar argument shows that it also satisfies (4.4).

That g satisfies (4.5) is obvious. To show that it satisfies (4.6), we observe that for $h = 0, 1, \ldots, k-2$,

$$\Delta^h \hat{g}(j) = i\Delta^h[\hat{f}(j)/j^*],$$

where $j^* = j$ if $j \neq 0$ and $0^* = 1$. By (3.7),

$$\Delta^h \hat{g}(j) = -\frac{1}{i} \sum_{s=0}^{h} \binom{h}{s} \Delta^s \hat{f}(j) \left[\Delta^{h-s} \frac{1}{(j+s)^*}\right];$$

hence, for $j \in A(f)$,

$$|\Delta^h \hat{g}(j)| \leqslant cB(f)|j|^{k-1-h-1}.$$

The second step is to show that m satisfies (4.7) for $h = k$. To do this we use (3.10) with $A = A(f)$. By (4.3) the left side is less than $B(f)$. Using the fact that $m \in l^\infty$, Lemma (3.1) and (4.4), the last term on the right is also majorized by $cB(f)$. For $h = 1, \ldots, k-1$,

$$\sum_{j\in A} \sum_{l \neq 0} \frac{\left|m(j+l) - \sum_{s=0}^{k-h-1} (l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2k-2h}} |\Delta^h \hat{f}_k(j)|^2 \leqslant cB(f),$$

by (4.6) and (4.7) for $h \leqslant k-1$. All of these and (4.5) yield (4.7) with $h = k$. This completes the proof of Lemma (4.1).

PROOF OF LEMMA (4.8). First we observe that it is sufficient to obtain a sequence of L^∞ functions, $\{g_{n,k}\}_{n=1}^{\infty}$, with the same properties as $\{f_n\}$. Then we can take f_n to be a partial sum of the Fourier series of $g_{n,k}$ such that $\int_{-\pi}^{\pi} |f_n|^2 |x|^{2k-1}\,dx \leqslant c\log n$, and $\hat{f}_n(j) = \hat{g}_{n,k}(j)$ for $|j| \leqslant (n+k)^r$.

The $g_{n,k}$'s are defined as follows. Let

$$g_{n,1}(x) = x^{-1}\left[\chi_{[1/n^2, 1/n]}(x) - \chi_{[1/n, 1]}(x)\right].$$

For $k = 2, 3, \ldots,$ let

$$g_{n,k}(x) = (e^{-ix} - 1)^{-1}\left[g_{n,k-1}(x) - e^{ix}g_{n,k-1}(2x)\right]. \tag{4.12}$$

Note that $g_{n,k}$ is bounded and is supported in $[2^{1-k}n^{-2}, 1]$.

We show that $\{g_{n,k}\}$ satisfies (4.2) by induction on k. It is obvious that $\hat{g}_{n,1}(0) = 0$. Suppose $\{g_{n,k-1}\}$, $k \geqslant 2$, satisfies (4.2). Let

$$\phi_{n,k}(x) = \frac{g_{n,k-1}(x)}{e^{-ix} - 1} = \sum_{j=-\infty}^{\infty} \hat{\phi}_{n,k}(j)e^{ijx}. \tag{4.13}$$

Then

$$\frac{e^{ix}g_{n,k-1}(2x)}{e^{-ix} - 1} = (1 + e^{ix}) \sum_{j=-\infty}^{\infty} \hat{\phi}_{n,k}(j)e^{i2jx}.$$

Therefore, (4.12) implies

$$\hat{g}_{n,k}(j) = \hat{\phi}_{n,k}(j) - \hat{\phi}_{n,,k}([j/2]). \tag{4.14}$$

It thus follows that $\hat{g}_{n,k}(0) = 0$. Also, from (4.13), we have

$$\hat{g}_{n,k-1}(j) = \hat{\phi}_{n,k}(j + 1) - \hat{\phi}_{n,k}(j).$$

Since $\hat{g}_{n,k-1}(j) = 0, j = 0, 1, \ldots, k - 2$, we have

$$\hat{\phi}_{n,k}(0) = \hat{\phi}_{n,k}(1) = \cdots = \hat{\phi}_{n,k}(k - 1),$$

so (4.14) implies $\hat{g}_{n,k}(j) = 0, j = 1, 2, \ldots, k - 1$. This proves (4.2). Inequality (4.4) with $B = c\log n$ can be verified by induction.

The proof of (4.5) and (4.6) is divided into two cases: $k = 1$ and $k > 1$. We first consider the case $k = 1$. Let r be a number between 1 and 2 to be chosen later. We write

$$\hat{g}_{n,1}(j) = \int_{n^{-2}}^{n^{-r}} \frac{1}{t}\,dt + \int_{n^{-2}}^{n^{-r}} \frac{1}{t}(e^{-ijt}-1)\,dt$$
$$+ \int_{n^{-r}}^{n^{-1}} \frac{1}{t}e^{-ijt}\,dt - \int_{n^{-1}}^{1} \frac{1}{t}e^{-ijt}\,dt.$$

The first term is equal to $(2 - r)\log n$. The modulus of the second term is less than $|j|n^{-r}$, which is less than 1 for $|j| < n^r$. The modulus of the third term is less than $(r - 1)\log n$. Upon integrating by parts, the modulus of the last term is less than $2n/|j|$, which is less than 2 for $|j| \geqslant n$. By picking $r = \frac{4}{3}$ and N sufficiently large, we obtain $|\hat{g}_{n,1}(j)| \approx c\log n$, $n \leqslant |j| < n^r$, $n \geqslant N$.

In order to simplify the proof of (4.5) and (4.6) for the case $k \geqslant 2$, we approximate $\{g_{n,k}\}$ by $\{h_{n,k}\}$, which is defined as follows. Let $h_{n,1}(x) = g_{n,1}(x)$. For $k = 2, 3, \ldots,$ let

$$h_{n,k}(x) = (-ix)^{-1}\left[h_{n,k-1}(x) - h_{n,k-1}(2x)\right].$$

Again, $h_{n,k}$ is bounded and is supported in $[2^{1-k}n^{-2}, 1]$. As in the proof of Lemma (2.5), we can, for $k > 1$, write

$$h_{n,k}(x) = \frac{1}{(-ix)^{k-1}} \sum_{j=0}^{k-1} c_{k,j}g_{n,1}(2^jx), \tag{4.15}$$

where $c_{k,j}$ is given by

$$\sum_{j=0}^{k-1} c_{k,j} x^j = \prod_{l=0}^{k-2} \left(1 - \frac{x}{2^l}\right). \tag{4.16}$$

It thus follows that

$$|h_{n,k}(x)| \leqslant \frac{c}{|x|^{k-1}} \sum_{j=0}^{k-1} |g_{n,1}(2^j x)|, \qquad k = 2, 3, \ldots.$$

For this and an induction on k, using the estimates

$$\left|\frac{1}{e^{-ix}-1} - \frac{1}{-ix}\right| \leqslant c, \qquad \left|\frac{e^{ix}}{e^{-ix}-1} - \frac{1}{-ix}\right| \leqslant c, \quad \text{for } |x| \leqslant \pi,$$

we obtain

$$|g_{n,k}(x) - h_{n,k}(x)| \leqslant \frac{c}{|x|^{k-2}} \sum_{j=0}^{k-1} |g_{n,1}(2^j x)|, \qquad k = 2, 3, \ldots. \tag{4.17}$$

To prove (4.5) for $g_{n,k}$ for $k > 2$, it is sufficient to show that

$$\left|\hat{h}_{n,k}(j) - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} \Delta^l \hat{h}_{n,k}(0)\right| \geqslant c|j|^{k-1} \log n, \qquad n \leqslant j \leqslant n^r, n \geqslant N, \tag{4.18}$$

where r, N are numbers to be chosen later. The reason is as follows. Since $\hat{g}_{n,k}(j) = 0, j = 0, 1, \ldots, k-2$,

$$\begin{aligned} |\hat{g}_{n,k}(j)| \geqslant &\left|\hat{h}_{n,k}(j) - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} \Delta^l \hat{h}_{n,k}(0)\right| \\ &- \left|(\hat{g}_{n,k} - \hat{h}_{n,k})(j) - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} \Delta^l (\hat{g}_{n,k} - \hat{h}_{n,k})(0)\right|. \end{aligned}$$

The second term on the right is equal to

$$\left|\int_{-\pi}^{\pi} [g_{n,k}(t) - h_{n,k}(t)] \left[e^{ijt} - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} (e^{-it} - 1)^l\right] dt\right|. \tag{4.19}$$

Using the fact that

$$\left|e^{-ijt} - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} (e^{-it} - 1)^l\right| \leqslant c|j|^{k-1} |t|^{k-1},$$

and (4.17), it is seen that (4.19) is majorized by $c\,|j\,|^{k-1}$. Thus, if N is large enough and (4.18) is assumed, we have

$$|\hat{g}_{n,k}(j)| \geqslant c|j|^{k-1} \log n, \qquad n \leqslant |j| \leqslant n^r, n \geqslant N.$$

In order to prove (4.18) we use (4.15) and a change of variables to obtain

$$\hat{h}_{n,k}(j) - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} \Delta^l \hat{h}_{n,k}(0)$$

$$= \sum_{s=0}^{k-1} c_{k,s} 2^{s(k-2)} \int_{-\pi}^{\pi} \frac{1}{(-it)^{k-1}} g_{n,1}(t) \left[e^{-ij2^{-s}t} - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} \left(e^{-i2^{-s}t} - 1\right)^l \right] dt$$

$$= \sum_{s=0}^{k-1} c_{k,s} 2^{s(k-2)} \int_{n^{-2}}^{n^{-r}} \frac{1}{(-it)^{k-1}} \frac{1}{t} \frac{j^{(k-1)}}{(k-1)!} \left(-it2^{-s}\right)^{k-1} dt$$

$$+ \sum_{s=0}^{k-1} c_{k,s} 2^{s(k-2)} \int_{n^{-2}}^{n^{-r}} \frac{1}{(-it)^{k-1}} \frac{1}{t}$$

$$\times \left[e^{-ij2^{-s}t} - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} (e^{-it2^{-s}} - 1)^l - \frac{j^{(k-1)}}{(k-1)!} \left(-i2^{-s}t\right)^{k-1} \right] dt$$

$$+ \sum_{s=0}^{k-1} c_{k,s} 2^{s(k-2)} \int_{n^{-r}}^{n^{-1}} \frac{1}{(-it)^{k-1}} \frac{1}{t} \left[e^{-ij2^{-s}t} - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} (e^{-it2^{-s}} - 1)^l \right] dt$$

$$- \sum_{s=0}^{k-1} c_{k,s} 2^{s(k-2)} \int_{n^{-1}}^{1} \frac{1}{(-it)^{k-1}} \frac{1}{t} \left[e^{-ij2^{-s}t} - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} (e^{-it2^{-s}} - 1)^l \right] dt.$$

By (4.16) the modulus of the first term is equal to

$$\prod_{l=1}^{k-1} \left(1 - \frac{1}{2^l}\right) \frac{|j^{(k-1)}|}{(k-1)!} (\log n)\,(2-r).$$

Using the estimate

$$\left| e^{-ij2^{-s}t} - \sum_{l=0}^{k-2} \frac{j^{(l)}}{l!} \left(e^{-i2^{-s}t} - 1\right)^l - \frac{j^{(k-1)}}{(k-1)!} \left(-i2^{-s}t\right)^{k-1} \right| \leqslant c|j|^k \left(t2^{-s}\right)^k,$$

the modulus of the second term is less than $c\,|j\,|^k n^{-r}$, which is less than $c\,|j\,|^{k-1}$ for $|j\,| < n^r$. A similar estimate shows that the modulus of the third term is less than

$$\sum_{s=0}^{k-1} |c_{k,s}| 2^{-s} \frac{|j^{(k-1)}|}{(k-1)!} (\log n)(r-1).$$

By estimating the integral of each term separately we obtain that the modulus of the last term is less than $c\,|j\,|^{k-1}$ for $|j\,| \geqslant n$. By choosing r close enough to 1 and N large enough we get (4.18).

A similar argument shows that in order to prove (4.6) for $k \geqslant 2$, it is sufficient to prove

$$\left| \Delta^h \hat{h}_{n,k}(j) - \sum_{l=0}^{k-h-2} \frac{j^{(l)}}{l!} \Delta^{h+l} \hat{h}_{n,k}(0) \right| \leqslant c|j|^{k-h-1} \log n, \tag{4.20}$$

$$n \leqslant |j| < n^r,\ n \geqslant N.$$

To prove (4.20) we observe that its left side is equal to

$$\int_{-\pi}^{\pi} h_{n,k}(t)(e^{-it}-1)^h\left[e^{-ijt}-\sum_{l=0}^{k-h-2}\frac{j^{(l)}}{l!}(e^{-it}-1)^l\right]dt.$$

This is less than

$$\sum_{s=0}^{k-1}|c_{k,s}|\int_{-\pi}^{\pi}|t|^{1-k}|g_{n,1}(2^st)|\,|t|^h|j|^{k-h-1}|t|^{k-h-1}\,dt \leqslant c|j|^{k-h-1}\log n,$$

by changing the variable and performing the integration. This concludes the proof of Lemma (4.8).

PROOF OF LEMMA (4.9). Define $f(x)=\Sigma_{k\leqslant|j|\leqslant N}e^{ijx}$. Then f has all the required properties.

We have thus completed the necessity part of the proof of Theorem (1.5).

5. Periodic case: sufficiency. Suppose $m\in l^\infty$, and, for $h=1,\ldots,k$, it satisfies (1.7). We use (3.9) and have

$$(5.1)\qquad \int_{-\pi}^{\pi}|(m\hat{f})^{\vee}|^2|x|^{2k-1}\,dx$$

$$\leqslant c\sum_j\sum_{l\neq0}\frac{|m(j+l)|^2\left|\hat{f}(j+l)-\Sigma_{s=0}^{k-1}(l^{(s)}/s!)\Delta^s\hat{f}(j)\right|^2}{l^{2k}}$$

$$+c\sum_{h=0}^{k-1}\sum_j\sum_{l\neq0}\frac{\left|m(j+l)-\Sigma_{s=0}^{k-h-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2k-2h}}|\Delta^h\hat{f}(j)|^2.$$

By the fact that $m\in l^\infty$ and Lemma (3.1), the first term is bounded by

$$c\int_{-\pi}^{\pi}|f|^2|x|^{2k-1}\,dx.$$

To estimate the second term, we let

$$w_{k,h}(j)=\sum_{l\neq0}\frac{\left|m(j+l)-\Sigma_{s=0}^{k-h-1}(l^{(s)}/s!)\Delta^s m(j)\right|^2}{l^{2k-2h}},\qquad h=0,1,\ldots,k-1.$$

Since $\hat{f}(l)=0$, $l=0,1,\ldots,k-1$,

$$|\Delta^h\hat{f}(j)|=\left|\int_{-\pi}^{\pi}f(t)(e^{-it}-1)^h\left[e^{-ijt}-\sum_{s=0}^{k-h-1}\frac{j^{(s)}}{s!}(e^{-it}-1)^s\right]dt\right|$$

$$\leqslant\int_{|t|\leqslant1/|j|}|f(t)|\,|t|^h|j|^{k-h}|t|^{k-h}\,dt$$

$$+\int_{1/|j|\leqslant|t|\leqslant\pi}|f(t)|\,|t|^h|j|^{k-h-1}|t|^{k-h-1}\,dt.$$

Therefore the last term of (5.1) is majorized by a sum of terms of the form

$$(5.2)\qquad c\sum_{|j|\geqslant1}|j|^{2k-2h}w_{k,h}(j)\left(\int_{|t|\leqslant1/|j|}|f(t)|\,|t|^k\,dt\right)^2$$

and

$$(5.3) \qquad c \sum_{|j| \geq 1} |j|^{2k-2h-2} w_{k,h}(j) \left(\int_{1/|j| \leq |t| \leq \pi} |f(t)| \, |t|^{k-1} \, dt \right)^2,$$

$h = 0, 1, \ldots, k-1$. By Hardy's inequality with weights [1], (5.2) is bounded by $\int_{-\pi}^{\pi} |f(x)|^2 |x|^{2k-1} \, dx$ provided we have

$$\sum_{1 \leq |j| \leq n} |j|^{2(k-h)} w_{k,h}(j) \frac{1}{n^2} \leq c.$$

This inequality is true since the left side is less than $\sum_{1 \leq |j| \leq n} |j|^{2(k-h-1)} w_{k,h}(j)$, which is less than c, by (1.7) with h replaced by $k-h$. Similarly (5.3) is majorized by $\int_{-\pi}^{\pi} |f(x)|^2 |x|^{2k-1} \, dx$ provided we have

$$\sum_{|j| \geq n} |j|^{2(k-h-1)} w_{k,h}(j) \log \pi n \leq c.$$

This is also true from (1.7) with h replaced by $k-h$. This concludes the proof of Theorem (1.5).

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Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada